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# Phase structure of interacting boson models in arbitrary dimension

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**Abstract**

We analyse the phase structure of a class of interacting boson models with two types of bosons, one scalar and one non-scalar, subject to one- and two-body interactions and with dynamic algebra $U(n)$. To these models, we associate a classical description in terms of $f = n - 1$ variables. We show that, if the system is invariant under two- or three-dimensional rotations (for $f \geqslant 2$ even or odd), the models have both first- and second-order phase transitions only if $f = 5, 9, 13, \ldots$. In the other $f \geqslant 2$ cases, the system has only second-order transitions. All phase transitions of this class of models belong to the cusp catastrophe in the classification of structurally unstable potentials.

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## 1. Introduction

In recent years, algebraic models based on boson realizations of the Lie algebra $U(n)$ have been shown to provide an effective description of many-body problems with $f = n-1$ degrees of freedom [1]. A particularly vast variety of applications of such models has been worked out in nuclear and molecular physics, where algebraic methods have been very successful in describing properties of collective rotational and vibrational spectra [2–6]. Applications to hadronic physics have also been presented [7] and applications to other areas have begun [8–10].

A common feature of these models is the occurrence of two 'phases' connected with specific configurations of the system's ground state: (i) a so-called 'spherical' phase characterized by the dynamic symmetry $U(f)$ and (ii) a $U(f)$-symmetry breaking phase, called 'deformed' phase, characterized by one or more remaining dynamic symmetries of the

  

**Table 1.** The $sb^{(l)}$ models with quasi-spin $l \leqslant 4$ and some of their applications. Not all models have had applications and models with isomorphic algebraic structure (for example, the Lipkin model and the 1D vibron model) are placed together.

| $l$ | $n$ | $f$ | Description and reference (if available) |
|---|---|---|---|
| $0(t)$ | 2 | 1 | Lipkin model:<br>a schematic many-body system [12]<br>1D vibron model:<br>stretching vibrations of molecules [3–5] |
| $\frac{1}{2}(\tau)$ | 3 | 2 | 2D vibron model:<br>bending vibrations in polyatomic molecules [6]<br>properties of cuprate superconductors [8, 9] |
| $1(p)$ | 4 | 3 | 3D vibron model:<br>rotations/vibrations of diatomic molecules [3–5] |
| $\frac{3}{2}(\pi)$ | 5 | 4 | |
| $2(d)$ | 6 | 5 | interacting boson model:<br>rotations/vibrations of atomic nuclei [2, 4, 5] |
| $\frac{5}{2}(\delta)$ | 7 | 6 | |
| $3(f)$ | 8 | 7 | |
| $\frac{7}{2}(\phi)$ | 9 | 8 | |
| $4(g)$ | 10 | 9 | |

model Hamiltonian. The crossover between the two phases gets sharper as the total number of bosons, $N$, increases and becomes of phase-transitional type in the $N \to \infty$ limit.

An important attribute of the asymptotic-$N$ transition is its order in the sense of Ehrenfest classification. The transition is said to be of the $k$th order if the $k$th derivative of the ground-state energy with respect to the relevant control parameter changes discontinuously at the phase-transitional point. While the first-order transition is associated with a sudden flip of the ground-state wavefunction, higher-order transitions are accompanied by more delicate structural changes. It turns out that the $U(f)$-symmetry breaking phase transitions in the above bosonic models are of the first or second order, the form of the phase diagram being in conformity with the classical Landau theory of phase transitions.

Phase transitions in interacting boson models have been extensively investigated ever since Gilmore [11] developed an algorithm for their study. The aim of this paper is two-fold: (a) to provide a general classification of the phase structure of this class of models, for an arbitrary classical dimension $f$ or, conversely, for any arbitrary quantum dimension, $n$, and (b) to associate this phase structure with the 'cusp' catastrophe in the analysis of structurally unstable potentials.

We consider here explicitly a class of algebraic boson models, with two types of bosons: a scalar boson, denoted by $s$, and another boson, denoted by $b^{(l)}$, with $f = 2l + 1$ components. The integer or half-integer number $l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ will be called the 'quasi-spin' of the $b$-boson. Use of both integer and half-integer $l$ allows one to treat within the same framework problems both in even- and odd-dimensional spaces ($f$ = even or odd). Problems on a line will be described by $l = 0$, in a plane by $l = \frac{1}{2}$, etc.

Several models belonging to this class, together with some of their applications, are shown in table 1. We emphasize the universal aspects of the applications in this table, ranging from cuprate superconductors to molecules and nuclei. Further potential applications are to atomic clusters [10] and atomic condensates.

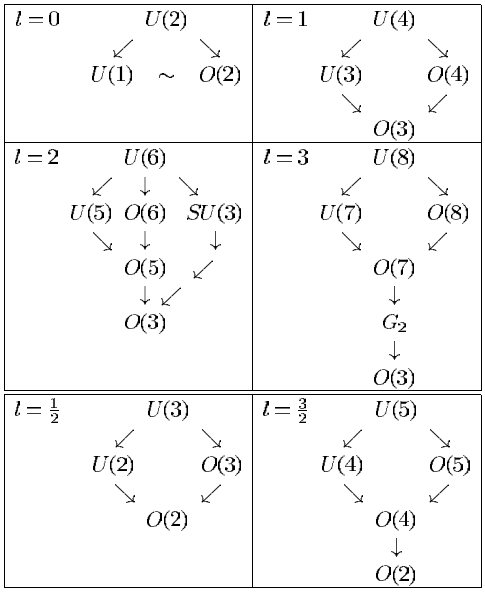


**Figure 1.** Dynamic-symmetry chains containing (for $n \geqslant 3$) the algebra of physical rotations, $O(3)$ or $O(2)$, for the lowest values of $l$. Note that the decomposition of $U(8)$ contains one of the exceptional algebras, $G_2$ [13].

## 2. Algebraic structure

Introducing boson creation and annihilation operators, $s^\dagger, s$ for $s$-bosons and $b_m^\dagger, b_m$ for $b$-bosons ($m = -l, \ldots, +l$), one can generate the Lie algebra of $U(n)$ composed of the bilinear products:

$$U(n) \doteqdot s^\dagger s, s^\dagger b_m, b_m^\dagger s, b_m^\dagger b_{m'} \qquad (m, m' = -l, \ldots, +l). \tag{1}$$

The Hamiltonian and transition operators are written in terms of the elements of $U(n)$. Also from the construction given above it is obvious that $n = f + 1$.

In order to determine the possible 'phases' of the system, one must decompose the algebra $U(n)$ into all its subalgebra chains with or without conditions. For applications in nuclear and molecular physics, it is convenient to impose three-dimensional (3D) rotational invariance for problems with $f =$ odd and two-dimensional (2D) rotational invariance for problems with $f =$ even. The case of $f = 1$ is special, since there is no rotational invariance to impose.

The decomposition of the algebra of $U(n)$, with the constraint that the angular momentum algebra $O(3)$ be contained in the chain, has been carried out explicitly for $n =$ even $= 2, 4, 6$. The decomposition is summarized in figure 1. For $n = 8, 10, \ldots$, it can be shown that at least two chains containing $O(3)$ always exist: (i) the chain $U(n) \supset U(n-1) \supset O(n-1)$ and (ii) the chain $U(n) \supset O(n) \supset O(n-1)$. These chains have in common $O(n-1)$, which has as a consequence that transitional Hamiltonians between $U(n-1)$ and $O(n)$ dynamic symmetries are integrable. It has been conjectured that no other chain exists under these circumstances, although no direct proof has been presented. The subalgebra chains of $U(8)$ are also shown in figure 1.

The decomposition of the algebra of $U(n)$ for $n =$ odd with the constraint the $O(2)$ be contained in the chain, has been carried out explicitly only for $n = 3$, and is shown in the

**Table 2.** Subalgebras of the $U(5)$ dynamic algebra associated with the $s\pi$-boson model ($l = \frac{3}{2}$). Note that the dynamic-symmetry chain $O(5) \supset O(4) \supset O(3) \supset O(2)'$, where $O(3) \doteqdot J_0, J_+, J_-$ and $O(2)' \doteqdot J_0$, is disregarded in the present interpretation of the model since it does not contain the physical $O(2) \doteqdot J_z$. Similarly, there exists another realization of $O(5)$ and an associated chain which does not conserve $J_z$.

Notation:

$b^\dagger_{\frac{k}{2}} \equiv \pi^\dagger_k, b_{\frac{k}{2}} \equiv \pi_k \quad (k = \pm 1, \pm 3)$

Operators:

$J_z = 3\pi^\dagger_{+3}\pi_{+3} + \pi^\dagger_{+1}\pi_{+1} - \pi^\dagger_{-1}\pi_{-1} - 3\pi^\dagger_{-3}\pi_{-3}$

$J_0 = \frac{1}{2}(-\pi^\dagger_{+3}\pi_{+3} + \pi^\dagger_{+1}\pi_{+1} - \pi^\dagger_{-1}\pi_{-1} + \pi^\dagger_{-3}\pi_{-3})$

$J_\pm = -\pi^\dagger_{\pm 3}\pi_{\pm 1} + \pi^\dagger_{\mp 1}\pi_{\mp 3}$

$K_\pm = \pi^\dagger_{\pm 3}\pi_{\mp 1} - \pi^\dagger_{\pm 1}\pi_{\mp 3}$

$P_k = \pi^\dagger_k s + s^\dagger \pi_{-k}$

Relevant subalgebras:

$U(4) \doteqdot \pi^\dagger_k \pi_{k'} \quad (k, k' = \pm 1, \pm 3)$

$O(5) \doteqdot J_z, J_0, J_+, J_-, K_+, K_-, P_{+3}, P_{+1}, P_{-1}, P_{-3}$

$O(4) \doteqdot J_z, J_0, J_+, J_-, K_+, K_-$

$O(2) \doteqdot J_z$

bottom part of figure 1. Since our interest is that of providing a general classification for arbitrary $n$, we have undertaken the explicit construction of the subalgebra chains for $n = 5$. This is also shown in figure 1. The study of the subalgebra chains required the explicit construction of the algebra itself (not done previously). This construction is shown in table 2. Although no applications exist of this boson model, it is of potential interest to problems in four-dimensional (4D) Euclidean spacetime.

## 3. Geometric analysis

### *3.1. Coset spaces*

Geometry can be associated to models with $U(n)$ algebraic structure by the introduction of the coset spaces $U(n)/U(n-1) \otimes U(1)$ [14]. There are $n-1$ complex coordinates in this space which we denote by $\alpha_m$. The coordinates $\alpha_m$ have the same transformation properties under $O(3)$ and $O(2)$ as the operators $b_m$. Properties of the ground state of the system can be obtained by introducing coherent states. There are various forms of coherent states, all related to each other. We use the projective coherent states that define a boson condensate [15–18]

$$|N\alpha\rangle = \frac{1}{\sqrt{N!}}\left(s^\dagger + \sum_m \alpha_m b^\dagger_m\right)^N |0\rangle. \tag{2}$$

Note that $\langle N\alpha|N\alpha\rangle = \mathcal{N}(\alpha)^N$, where $\mathcal{N}(\alpha) = 1 + \sum_m |\alpha_m|^2$.

The energy expectation value in states (2) reads as $\langle N\alpha|H|N\alpha\rangle/\langle N\alpha|N\alpha\rangle \equiv E(\alpha)$, where $H$ is the Hamiltonian of the system. Minimization of $E(\alpha)$ in parameters $\alpha$ yields a variational ground state. The nature of phase transitions in interacting boson models can be studied by taking the $N \to \infty$ limit, in which the variational ground state becomes exact and may exhibit nonanalytic changes with external parameters.

*3.2. Even n (integer l)*

For integer quasi-spin $l$, we require invariance of the system under 3D rotations. This is best imposed by introducing spherical tensors with respect to $O(3)$. If the boson creation operators $b^\dagger_m$ transform as tensors of rank $l$, the corresponding annihilation operators do not. Spherical tensors can be constructed by introducing the operators $\tilde{b}_m = (-)^{l-m} b_{-m}$. Tensor products of operators are introduced in the usual way. We use in this paper the notation

$$[TU]^{(L)}_M = \sum_{m,m'} (lml'm'|LM) T_m U_{m'} \tag{3}$$

for a tensor product of a tensor of rank $l$ by a tensor of rank $l'$. The coefficient in equation (3) is a Clebsch–Gordan coefficient of $O(3)$. Written in this notation, the angular momentum operators of the presently considered class of boson models read as

$$J_M \sim [b^\dagger \tilde{b}]^{(1)}_M \qquad (M = 0, \pm 1). \tag{4}$$

The most general rotationally invariant $sb^{(l)}$-interacting boson model Hamiltonian with one- and two-body interactions can be written as follows:

$$\begin{aligned} H = {} & \epsilon_0 + e_s s^\dagger s + e_b [b^\dagger \tilde{b}]^{(0)} + e'_s s^\dagger s^\dagger ss + u[[b^\dagger s^\dagger][\tilde{b}s]]^{(0)} + u'([b^\dagger b^\dagger]^{(0)} ss + \text{H.c.}) \\ & + v([[b^\dagger b^\dagger]^{(l)} [\tilde{b}s]]^{(0)} + \text{H.c.}) + \sum_L w_L [[b^\dagger b^\dagger]^{(L)} [\tilde{b}\tilde{b}]^{(L)}]^{(0)}, \end{aligned} \tag{5}$$

where $\epsilon_0, e_s, e_b, e'_s, u, u', v$, and $w_L$ ($L = 0, 2, \ldots, 2l$) are arbitrary constants. We have written Hamiltonian (5) in the normal-ordered form, in order to classify all possible terms. Non-normal ordered interactions, such as

$$[[b^\dagger \tilde{b}]^{(l)} [b^\dagger s]]^{(0)} + \text{H.c.},$$

can be obtained from the normal-ordered form by $O(3)$ recoupling and need not be considered separately.

Since in the class of models discussed here, the single-particle space is finite, having dimension $n$, the contribution of one-body terms to the ground-state energy grows as $N$, while that of two-body terms grows as $N(N-1)$. It is convenient, for the study of phase transitions in these systems, to consider the energy per particle, and to scale two-body terms with an extra factor of $(N-1)$ relative to the one-body terms [18]. Therefore, we will use the scaling

$$\begin{aligned} & (e_s, e_b) \to \frac{1}{N}(e_s, e_b), \\ & (e'_s, u, u', v, w_L) \to \frac{1}{N(N-1)}(e'_s, u, u', v, w_L). \end{aligned} \tag{6}$$

The average energy per particle corresponding to the Hamiltonian of equation (5) in coherent states (2) is

$$\begin{aligned} E(\alpha) = {} & \epsilon_0 + e_s \frac{1}{\mathcal{N}(\alpha)} + e_b \frac{[\alpha^* \tilde{\alpha}]^{(0)}}{\mathcal{N}(\alpha)} + e'_s \frac{1}{\mathcal{N}(\alpha)^2} + u \frac{[\alpha^* \tilde{\alpha}]^{(0)}}{\mathcal{N}(\alpha)^2} + u' \frac{[\alpha^* \alpha^*]^{(0)} + \text{c.c.}}{\mathcal{N}(\alpha)^2} \\ & + v \frac{[[\alpha^* \alpha^*]^{(l)} \tilde{\alpha}]^{(0)} + \text{c.c.}}{\mathcal{N}(\alpha)^2} + \sum_L w_L \frac{[[\alpha^* \alpha^*]^{(L)} [\tilde{\alpha}\tilde{\alpha}]^{(L)}]^{(0)}}{\mathcal{N}(\alpha)^2}, \end{aligned} \tag{7}$$

where $\tilde{\alpha}_m = (-)^{l-m} \alpha_{-m}$. Note that $E(\alpha)$ does not depend on $N$, and that it represents, in the limit $N \to \infty$, the classical Hamiltonian of the system. This is so because of scaling (6), which implies that quantum fluctuations of the energy per particle,

$$\frac{\langle N\alpha | H^2 | N\alpha \rangle}{\langle N\alpha | N\alpha \rangle} - \left( \frac{\langle N\alpha | H | N\alpha \rangle}{\langle N\alpha | N\alpha \rangle} \right)^2,$$

die out as $\mathcal{O}(N^{-1})$. In fact, the value of $N^{-1}$ plays the role of Planck constant here. This is an important distinction of the present class of models, with a finite-dimensional single-particle Hilbert space, from infinite models, e.g., the lattice models, for which quantum fluctuations disappear only at critical values of the interaction strengths indicating quantum phase transitions [19]. In our case, 'quantum' phase transitions discussed below result solely from the classical properties of the system.

The variables $\alpha_m$ are abstract variables in the coset space. It is possible to introduce canonical coordinates $q_m$ and momenta $p_m$ by [20]

$$q_m = \frac{(-)^m \alpha_{-m} + \alpha_m^*}{\sqrt{2}}, \qquad p_m = \mathrm{i}\frac{(-)^m \alpha_{-m}^* - \alpha_m}{\sqrt{2}}, \tag{8}$$

which have the property $q_m^* = (-)^m q_{-m}$ and $p_m^* = (-)^m p_{-m}$. The new variables are still complex, but due to their $m \to -m$ symmetry one can separate $(2l+1)$ real coordinates (e.g., real and imaginary parts of $q_m$ with $m \geqslant 0$) and the same number of associated real momenta.

For the study of 'spherical' to 'deformed' phase transitions reported in this paper, it is further convenient to introduce hyperspherical coordinates. Real variables $\mathrm{Re}\, q_m$ and $\mathrm{Im}\, q_m$ (with $m \geqslant 0$) are expressed as

$$\begin{aligned}
q_0 &= \beta \cos\theta_1, \\
\sqrt{2}\,\mathrm{Im}\, q_{+1} &= \beta \sin\theta_1 \cos\theta_2, \\
\sqrt{2}\,\mathrm{Re}\, q_{+1} &= \beta \sin\theta_1 \sin\theta_2 \cos\theta_3, \\
&\vdots \\
\sqrt{2}\,\mathrm{Im}\, q_{+l} &= \beta \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{2l-1} \cos\theta_{2l}, \\
\sqrt{2}\,\mathrm{Re}\, q_{+l} &= \beta \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{2l-1} \sin\theta_{2l},
\end{aligned} \tag{9}$$

where the radius $\beta$ measures the overall degree of deformation ($\beta = 0$ for the 'spherical' phase and $\beta > 0$ for the 'deformed' phase) while the angles $\theta_i$ ($i = 1, 2, \ldots, 2l$) $\equiv \Theta$ determine the deformed shape type and orientation.

It is worth noting that some of the angles in equation (9) can be eliminated by an appropriate rotation of the coordinate system (the Hamiltonian being rotationally invariant). This elimination is simple for $l = 1, 2$, but becomes complicated (if at all possible [21]) for $l = 3, 4, \ldots$. In the following general discussion, we will keep all $2l$ hyperspherical angles defined in equation (9).

The analysis of the phase structure of interacting boson model proceeds then as follows: we set in the classical Hamiltonian all momenta $p_m = 0$ and consider the potential energy per particle $V(q) \equiv E(q, p = 0)$. This functional, when expressed in hyperspherical coordinates, has the form

$$V(\beta, \Theta) = \frac{A\beta^2}{1+\beta^2} + \frac{A'\beta^2}{(1+\beta^2)^2} + \frac{B(\Theta)\beta^3}{(1+\beta^2)^2} + \frac{C(\Theta)\beta^4}{(1+\beta^2)^2}, \tag{10}$$

where $A = e_b/\sqrt{2l+1}$, $A' = [u + 2(-)^l u']/\sqrt{2l+1}$, while the coefficients $B$ and $C$ are discussed below. We set $\epsilon_0 = e_s = e_s' = 0$ to avoid unimportant terms of zeroth power in $\beta$.

Concrete forms of functions $B(\Theta)$ and $C(\Theta)$ from equation (10) must be evaluated specifically for each given quasi-spin $l$ using the following formulae:

$$\begin{aligned}
B(\Theta) &= \frac{1}{\sqrt{2}} \frac{v}{\sqrt{2l+1}} \sum_M (-)^M \mathrm{Re}\big([qq]_M^{(l)} q_{-M}\big)_{\beta=1}, \\
C(\Theta) &= \frac{1}{4} \sum_L \frac{w_L}{\sqrt{2L+1}} \sum_M \big|[qq]_M^{(L)}\big|^2_{\beta=1}
\end{aligned} \tag{11}$$

**Table 3.** Coefficients $A$, $A'$, $B$ and $C$ in the potential energy functional for the lowest values of $l$.

| l | Coefficients |
|---|---|
| 0 | $A = e_b$<br>$A' = u + 2u'$<br>$B = \frac{v}{\sqrt{2}}$<br>$C = \frac{w_0}{4}$ |
| 1 | $A = \frac{e_b}{\sqrt{3}}$<br>$A' = \frac{u-2u'}{\sqrt{3}}$<br>$B = 0$<br>$C = \frac{1}{3}(\frac{w_0}{4} + \frac{w_2}{2\sqrt{5}})$ |
| 2 | $A = \frac{e_b}{\sqrt{5}}$<br>$A' = \frac{u+2u'}{\sqrt{5}}$<br>$B = -\frac{v}{\sqrt{35}}\cos 3\theta_1$<br>$C = \frac{1}{5}(\frac{w_0}{4} + \frac{\sqrt{5}w_2}{14} + \frac{3w_4}{14})$ |
| $\frac{1}{2}$ | $A = \frac{e_\pm}{2}$<br>$A' = \frac{u_\pm}{2}$<br>$B = 0$<br>$C = \frac{w_{\pm\pm\pm\pm}+w_{\pm\mp\pm\mp}}{4}$ |
| $\frac{3}{2}$ | $A = \frac{e_{\pm1}}{2}(\cos^2\theta_1 + \sin^2\theta_1\cos^2\theta_2) + \frac{e_{\pm3}}{2}\sin^2\theta_1\sin^2\theta_2$<br>$A' = \frac{u_{\pm1}}{2}(\cos^2\theta_1 + \sin^2\theta_1\cos^2\theta_2) + \frac{u_{\pm3}}{2}\sin^2\theta_1\sin^2\theta_2$<br>$B = 0$<br>$C = \frac{w_{\pm3\pm3\pm3\pm3}+w_{\pm3\mp3\pm3\mp3}}{8}\sin^4\theta_1\sin^4\theta_2$<br>$+ \frac{w_{\pm1\pm1\pm1\pm1}+w_{\pm1\mp1\pm1\mp1}}{8}(\cos^2\theta_1 + \sin^2\theta_1\cos^2\theta_2)^2$<br>$+ \frac{w_{\pm3\pm1\pm3\pm1}+w_{\pm3\mp1\pm3\mp1}-w_{\pm3\mp3\pm1\mp1}}{2}$<br>$\times \sin^2\theta_1\sin^2\theta_2(\cos^2\theta_1 + \sin^2\theta_1\cos^2\theta_2)$ |

The list of these functions for the lowest values of $l$ is given in table 3. For $l = 0$, there is no angle dependence (trivially). For $l = 1$ the angle dependence can be eliminated completely by rotation of the axes. For $l = 2$, there is dependence on one angle $\theta_1$ (the other three can be eliminated, set to $\theta_2 = \theta_3 = \theta_4 = \frac{\pi}{2}$, by rotation). The angle $\theta_1$ is identified with the Bohr angle $\gamma$ used in nuclear physics in the description of quadrupole oscillations of liquid drops [22]. The constancy of $C(\Theta)$ results from the fact that for $l = 2$ there exist only two independent scalar couplings of coordinates—those given by quadratic and cubic terms—so that the quartic term in equation (7) is just proportional to the squared quadratic term [23].

For $l = 3, 4, \ldots$, the explicit construction of the potential $V(\beta, \Theta)$ becomes increasingly difficult. Nonetheless, some generic statements can be made. For $l =$ odd, the cubic term is zero. This is due to the fact that Bose symmetry implies that two boson creation or two boson annihilation operators can be coupled only to even values of the angular momentum $L = 0, 2, 4, \ldots$. This even angular momentum, $L$, cannot in turn be coupled with odd $l$, to give zero. Thus the term $[[b^\dagger b^\dagger]^{(l)}[\tilde{b}s]]^{(0)}$ + H.c. in Hamiltonian (5) is identically zero. For $l =$ even, instead, a cubic term can be constructed. This property distinguishes even from odd cases.

### *3.3. Odd n (half-integer l)*

As explained in section 2, for half-integer quasi-spins $l$ of the $b$-boson we relax the requirement upon rotational invariance of the Hamiltonian. Instead of the full invariance under 3D rotations we only assume $O(2)$ invariance under 2D rotations, generated by

$$J_z = 2\sum_m m b_m^\dagger b_m. \tag{12}$$

The most general 2D rotational invariant one- and two-body Hamiltonian is

$$\begin{aligned} H = {} & \epsilon_0 + e_s s^\dagger s + \sum_m e_m b_m^\dagger b_m + e'_s s^\dagger s^\dagger s s + \sum_m u_m b_m^\dagger s^\dagger b_m s + \sum_m u'_m \left(b_m^\dagger b_{-m}^\dagger s s + \text{H.c.}\right) \\ & + \sum_{\substack{m_1+m_2=\\ m'_1+m'_2}} w_{m_1 m_2 m'_1 m'_2} \left(b_{m_1}^\dagger b_{m_2}^\dagger b_{m'_1} b_{m'_2} + \text{H.c.}\right), \end{aligned} \tag{13}$$

where $\epsilon_0, e_s, e'_s, e_m, u_m, u'_m$, and $w_{m_1 m_2 m'_1 m'_2}$ (with $m, m_1, m'_1, m_2, m'_2 = -l, \ldots, +l$) are interaction constants.

The coherent-state procedure outlined in section 3.2, in particular equation (2), is again employed, yielding the following energy functional:

$$\begin{aligned} E(\alpha) = {} & \epsilon_0 + e_s \frac{1}{\mathcal{N}(\alpha)} + \sum_m e_m \frac{|\alpha_m|^2}{\mathcal{N}(\alpha)} + e'_s \frac{1}{\mathcal{N}(\alpha)^2} + \sum_m u_m \frac{|\alpha_m|^2}{\mathcal{N}(\alpha)^2} \\ & + \sum_m u'_m \frac{\alpha_m^* \alpha_{-m}^* + \text{c.c.}}{\mathcal{N}(\alpha)^2} + \sum_{\substack{m_1+m_2=\\ m'_1+m'_2}} w_{m_1 m_2 m'_1 m'_2} \frac{\alpha_{m_1}^* \alpha_{m_2}^* \alpha_{m'_1} \alpha_{m'_2} + \text{c.c.}}{\mathcal{N}(\alpha)^2}. \end{aligned} \tag{14}$$

Variables $\alpha_m$ and $\alpha_m^*$ must all together define $(2l+1)$ classical coordinates and the same number of associated momenta. To ensure the continuity with the integer-$l$ case, equation (8) can again be used, yielding the relation $q_{-m} = (-)^m q_m^*$ between coordinate components with positive and negative values of $m$. The $m > 0$ components are then expressed in the hyperspherical parametrization,

$$\begin{aligned} \sqrt{2}\,\text{Im}\, q_{+\frac{1}{2}} &= \beta \cos\theta_1, \\ \sqrt{2}\,\text{Re}\, q_{+\frac{1}{2}} &= \beta \sin\theta_1 \cos\theta_2, \\ &\vdots \\ \sqrt{2}\,\text{Re}\, q_{+l} &= \beta \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{2l} \end{aligned} \tag{15}$$

(cf equation (9)).

The potential energy (with $\epsilon_0 = e_s = e'_s = 0$) has the following form now:

$$V(\beta, \Theta) = \frac{A(\Theta)\beta^2}{1+\beta^2} + \frac{A'(\Theta)\beta^2}{(1+\beta^2)^2} + \frac{C(\Theta)\beta^4}{(1+\beta^2)^2}, \tag{16}$$

where

$$\begin{aligned} A(\Theta) &= \frac{1}{2}\sum_m e_m |q_m|^2_{\beta=1}, \qquad A'(\Theta) = \frac{1}{2}\sum_m u_m |q_m|^2_{\beta=1}, \\ C(\Theta) &= \frac{1}{2}\sum_{\substack{m_1+m_2=\\ m'_1+m'_2}} w_{m_1 m_2 m'_1 m'_2}\, \text{Re}[q_{m_1} q_{m_2} q_{m'_1}^* q_{m'_2}^*]_{\beta=1}. \end{aligned} \tag{17}$$

Note that the terms with $u'_m$ do not appear here since the corresponding interactions contribute only to the kinetic energy.

Concrete values of coefficients from equation (16) for the lowest values of $l$ are given in the second part of table 3. Apparently, the angular dependence is more complicated than in the integer-$l$ case, which follows from the lower symmetry of Hamiltonian (13). In fact, to simplify the formulae in table 3 we have additionally assumed that the system is invariant

under $z \to -z$ reflection, so that $e_m = e_{-m}$, $w_{m_1 m_2 m_1' m_2'} = w_{-m_1 -m_2 -m_1' -m_2'}$ etc. One can notice that, for $l = \frac{1}{2}$, functions do not depend on angles (the single angle $\theta_1$ can be eliminated by 2D rotations), while for $l = \frac{3}{2}$ depend only on two angles (the third one being eliminated by 2D rotations). Also, in the case in which the single-particle energies and interaction strengths do not depend on $m$, the angle dependence for $l = \frac{3}{2}$ disappears.

A generic feature of all half-integer $l$ cases is the absence of a cubic term in the potential $V(\beta, \Theta)$. This result has its roots in the 2D rotational invariance of the Hamiltonian. For half-integer values it is not possible to satisfy the condition $m_1 + m_2 + m_3 = 0$. The absence of cubic terms has dramatic consequences on the nature of the phase transitions, as discussed in the following sections.

## 4. Order of phase transitions

Given the potential energy functionals (10) or (16), the analysis of phase transitions reduces to the usual classical analysis (a) minimizing the potential functional and (b) looking for places in the parameter space where the minimum evolves in a nonanalytic way. The equilibrium values $(\beta_0, \Theta_0)$ can be taken as classical order parameters. As the number of dimensions increases there are many phase transitions between different geometric configurations, mostly in angle variables. In this paper, we are interested in phase transitions in the variable $\beta$, usually called the 'deformation' variable.

Corresponding to these classical phase transitions, there are also 'quantum' phase transitions. The order parameters here are the ground-state expectation values of some operators. A commonly used order parameter for the phase transitions we are interested in is the ground-state expectation value of the number operator for $b$-bosons, $\langle n_b \rangle_0$. The so-called 'spherical' phase has $\langle n_b \rangle_0 = 0$, while the so-called 'deformed' phase has $\langle n_b \rangle_0 > 0$. Classical and quantum order parameters are related via the expression

$$\langle n_b \rangle_0 = \frac{N\beta_0^2}{1 + \beta_0^2}. \tag{18}$$

The 'quantum' phase transitions in the present sense are closely related to quantum phase transitions studied in infinite lattice and many-body systems [19], although, as explained above, interacting boson models with a finite single-particle space yield purely classical behaviour in the asymptotic-$N$ limit. In this paper, we do not discuss scaling behaviour and divergence of correlations, which are landmarks of quantum phase transitions in infinite models, since for the class of models presented here they have been discussed recently by Vidal *et al* [24] and Rowe *et al* [25]. The term 'phase transition' is used here solely in the sense of a nonanalytic change of the equilibrium (ground-state) configuration.

In order to study the phase structure of interacting boson models, we consider a family of Hamiltonians (5) or (13) with the interaction strengths being some specific smooth functions of a single external parameter, denoted here as $t$. Without the loss of generality we assume that the dependence on $t$ is such that the sum $a = A + A'$ of coefficients of quadratic terms in equations (10) or (16) increases from $a \ll 0$ to $a \gg 0$, which means that we proceed from 'deformed' to the 'spherical' phase.

The analysis is rather simple for $l = 0, \frac{1}{2}, 1$, when the potentials (10) and (16) do not depend on angles. The case of $l = 2$ is special and has been extensively investigated [2, 16–18]. One exploits the fact that $C(\Theta)$ does not depend on angles (see table 3). This leads to the potential energy functional

$$V(\beta, \gamma; t) = \frac{a(t)\beta^2 + b(t)\beta^3 \cos 3\gamma + c(t)\beta^4}{(1 + \beta^2)^2}, \tag{19}$$

where $\beta, \gamma$ are Bohr variables [22], while the coefficients $a, b, c$ are related to those in table 3 via simple expressions $a(t) = A(t)+A'(t)$, $b(t) = -v(t)/\sqrt{35}$ and $c(t) = A(t)+C(t)$.

Negative values of $c$ are feasible only in a certain range of the parameter space which does not overlap with the phase-transitional region, so from the present point of view $c(t)$ may be regarded as a positive scaling constant. Furthermore, the minimization of function (19) in variable $\gamma$ can be performed separately, yielding either $\gamma_0 = 0$ or $60^\circ$, and to find an equilibrium value $\beta_0 \geqslant 0$ of $\beta$ we just need to minimize the simplified form

$$V'(\beta; t) = \frac{a'(t)\beta^2 + b'(t)\beta^3 + \beta^4}{(1+\beta^2)^2}, \tag{20}$$

where $a'(t) = a(t)/c(t)$, and $b'(t) = -|b(t)|/c(t) \leqslant 0$.

Equation (20) exhibits the well-known phase properties [17, 18], namely the first-order phase separatrix

$$(b')^2 - 4a' = 0, \quad b' < 0 \qquad \text{(1st order)} \tag{21}$$

between 'spherical' and 'deformed' equilibrium shapes (here, the coexisting $\beta_0 = 0$ and $\beta_0 > 0$ minima exchange their roles of local and global equilibria) with the $b' = 0$ endpoint of a second-order transition (where $\beta_0 = 0$ local maximum absorbs $\beta_0 > 0$ minimum and turns into a new minimum). On the deformed side of the phase diagram, close to the second-order transition, the deformation parameter $\beta_0$ behaves as a square root of the distance from the critical point measured by an increment of $a'$; hence the critical exponent is $\lambda = 1/2$.

For integer $l \geqslant 3$ and half-integer $l \geqslant \frac{3}{2}$, however, the general phase-transitional analysis is hindered by a complex dependence of the respective potential energy functionals on $\Theta$. In fact, the number of essential control parameters of the problem is generally larger than two because the functions $A(\Theta)$, $A'(\Theta)$, $B(\Theta)$ and $C(\Theta)$ contain several arbitrary interaction strengths. Moreover, the potential energy form may not allow separation of the minimization procedures that determine the equilibrium values $\beta_0$ and $\Theta_0$.

In spite of these difficulties, it is still possible to formulate, for integer $l \geqslant 3$ and half-integer $l \geqslant \frac{3}{2}$, a rescaled problem (20), where coefficients

$$a'(t) = \frac{A(t) + A'(t)}{|A(t) + C(t, \Theta_0(t))|}, \qquad b'(t) = \frac{B(t, \Theta_0(t))}{|A(t) + C(t, \Theta_0(t))|} \tag{22}$$

are derived from dependences $A(t)$, $A'(t)$, $B(t, \Theta)$, $C(t, \Theta)$, and from angles $\theta_{0i}(t) (i = 1, \ldots, 2l) \equiv \Theta_0(t)$ obtained by the overall minimization (involving both $\Theta$ and $\beta$ variables) of energy functional (10) or (16). The sign at $\beta^4$ is considered positive for the reason explained above. It is clear that the equilibrium value $\beta_0(t) \geqslant 0$ must minimize $V'(\beta; t)$. If $a'(t)$ and $b'(t)$ happen to be such that $\beta_0(t) > 0$, the system is in the 'deformed phase'. On the other hand, if $\beta_0(t) = 0$, the system is 'spherical' and the angles $\Theta_0$ turn irrelevant.

A simple analysis of the form (20) reveals the known curve (21) of first-order phase transitions ended at $b' = 0$ by the second-order critical point. However, in the case $l =$ even $\geqslant 4$ the second-order phase separatrix extends to the entire half-line

$$a' = 0, \quad b' \geqslant 0 \qquad \text{(2nd order)}. \tag{23}$$

This can be obtained by the $\beta \to -\beta$ inversion of the $b' \leqslant 0$ situation keeping in mind that only the $\beta \geqslant 0$ solutions are physical. Remind that the domain (23) is contracted to a single critical point in the $l = 2$ case because the sign of $b'$ (resulting from the separate minimization of the $\propto \cos 3\gamma$ term in $\gamma$) cannot be positive. However, in the most general situation with $\beta_0$ and $\Theta_0$ entangled neither sign of $b'$ can be *a priori* excluded. As a result,

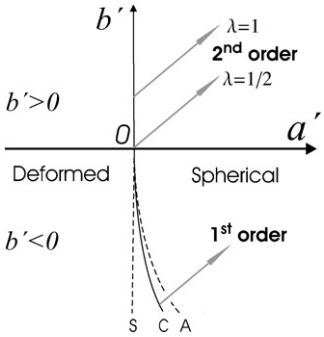


**Figure 2.** Schematic phase diagram corresponding to equation (20). The full curve (C) indicates the first- or second-order phase separatrix, while the dashed curves correspond to spinodal (S) and antispinodal (A) lines demarcating the phase-coexistence region.

**Table 4.** Summary of ground-state 'deformed-to-spherical' phase transitions in the $sb^{(l)}$-boson models with $l \leqslant 4$. The case of $l = 0$ is exceptional and is discussed in section 5.

| $l$ | Spherical-deformed phases | Types of phase transitions |
|---|---|---|
| 0 | $U(1) - O(2)$ | 1st order, 2nd order $\lambda = 1/2$ |
| 2 | $U(5) - O(6)$, $SU(3)$ | |
| 4 | $U(9) - O(10), \ldots$ | 1st order, 2nd order $\lambda = 1/2$ |
| ⋮ | ⋮ | 2nd order $\lambda = 1$ possible |
| 1 | $U(3) - O(4)$ | 2nd order $\lambda = 1/2$ |
| 3 | $U(7) - O(8)$ | |
| ⋮ | ⋮ | |
| $\frac{1}{2}$ | $U(2) - O(3)$ | 2nd order $\lambda = 1/2$ |
| $\frac{3}{2}$ | $U(4) - O(5)$ | |
| $\frac{5}{2}$ | $U(6) - O(7)$ | |
| $\frac{7}{2}$ | $U(8) - O(9)$ | |
| ⋮ | ⋮ | |

second-order spherical-deformed phase transitions may in principle appear also away from the $[U(n-1) - O(n)] \supset O(n-1)$ transitional path (characterized by $b' = 0$). While at $b' = 0$ the critical exponent in deformation parameter is $\lambda = 1/2$, in the $b' > 0$ case we have $\lambda = 1$. The most general phase diagram in the plane $a' \times b'$ is depicted in figure 2.

Figure 2 represents the generic situation. However, the analysis of the previous section shows that for reasons of invariance under $O(3)$ or $O(2)$, cubic terms are missing in the potential energy functional for integer $l$ = odd and for half-integer $l$. Because the cubic term is necessary for creating first-order transitions as well as the second-order transitions with $\lambda = 1$, the present observation implies that bosons with odd- and half-integer values of $l$ only allow for second-order $\lambda = 1/2$ transitions between 'deformed' and 'spherical' shapes. A summary of results for the lowest values of $l$ is given in table 4.

The above discussion was restricted to phase transitions in the $\beta$ variable. Phase transitions in angle variables may take place [26] at some values of $t$. The classification of phase transitions in angle variables is still an open problem.

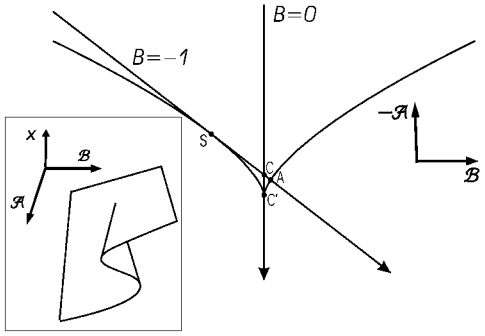


**Figure 3.** Mapping (27) of $B = -1$ and $B = 0$ trajectories with $A \in (-\infty, +\infty)$ from the parameter space of equation (26) to the cusp form (24). The cusp topology is shown in the inset. The spinodal, critical and antispinodal points of the 'deformed-to-spherical' evolution correspond to points S, C and A, respectively (C' for $B = 0$). At the spinodal point, the position of the $\beta_0 < 0$ minimum reaches zero and becomes physical.

## 5. Relation to the cusp catastrophe

It is known that a useful classification of the mean-field phase transitions can be obtained from the analysis of structurally unstable potentials within the framework of the catastrophe theory [27, 28]. One of the most common catastrophes is the one with germ $x^4$ and codimension equal to 2, called the cusp. Its potential reads as

$$\mathcal{V}(x) = x^4 + \mathcal{A}x^2 + \mathcal{B}x, \tag{24}$$

where $x$ is a coordinate and $\mathcal{A}$, $\mathcal{B}$ two control parameters. Within a cusp-like region

$$\mathcal{A} < 0, \qquad |\mathcal{B}| \leqslant \frac{4}{3\sqrt{6}}(-\mathcal{A})^{3/2} \tag{25}$$

the potential develops a bistable form with two coexisting minima at $x > 0$ and $x < 0$ (see figure 3(inset)). If $\mathcal{B}$ crosses zero the minima swap, giving rise to a first-order structural phase transition.

The theory shows that all catastrophes with two control parameters and one dynamical variable are topologically equivalent to the cusp [27, 28]. Indeed, phase diagrams of a vast variety of systems are known to have similar forms resulting from the inherent connection with the cusp catastrophe [29–31].

Also the behaviours discussed in section 4 belong to this class. In order to show this explicitly, we need to transform the rescaled potential energy (20) to the canonical cusp form (24). Since the denominator in equation (20) plays only the role of a form factor, it can be removed, within a physically relevant interval $\beta \in [0, \beta_{\max}]$, by a smooth and monotonous transformation $\beta \to \bar{\beta} = f(\beta)$ (Taylor expansion of $f^{-1}$ can be obtained term by term using the expanded form of equation (20)). Therefore, let us consider a transformed interacting boson model potential

$$V(\bar{\beta}) = \bar{\beta}^4 + A\bar{\beta}^2 + B\bar{\beta}^3, \tag{26}$$

where $A$ and $B$ now stand for some redefined control parameters. It is easy to see that a continuous mapping $(A, B) \mapsto (\mathcal{A}, \mathcal{B})$ from equation (26) to (24) can be achieved via appropriate parameter-dependent shifts of $x$ and $\mathcal{V}$, yielding simple relations

$$\mathcal{A} = A - \tfrac{3}{8}B^2, \qquad \mathcal{B} = \tfrac{1}{2}B\left(\tfrac{1}{4}B^2 - A\right). \tag{27}$$

In spite of the simplicity of the above analysis, there exists an obstacle to a direct comparison of forms (24) and (26). It is the fact that for $l > 0$ the potential energy minima with negative $\beta_0$ are not relevant ($\beta$, as a radius, must be semipositive). That is why, as follows from equation (27), a first-order phase transitional path with $B = \text{const} < 0$ and $A \in (-\infty, +\infty)$ in the IBM parameter plane maps onto a tilted line in figure 3. The line is located within the cusp region (25) all the way until the coefficient $A$ reaches the antispinodal value, $A = \frac{9}{32}B^2$ (point A), but before the spinodal value, $A = 0$ (point S), one of the cusp minima corresponds to a nonphysical value $\beta_0 < 0$. It is only between points S and A where the potential energy exhibits real phase coexistence form, the swap of both minima taking place at C. Similarly, the second-order phase transitional path with $B = 0$ is mapped onto a vertical line in figure 3 crossing the cusp point when $A = 0$, i.e., at the critical point. The second-order phase transitional path with $B = \text{const} > 0$ (not shown in the figure) is associated with a mirror conjugate of the $B < 0$ path with respect to the vertical axis, the critical point corresponding to the intersection with the left cusp boundary.

It is worth making here a comment on the exceptional situation with $l = 0$. In this case, the configuration space is one-dimensional and both signs of $\beta$ are allowed. The potential energy has two *real* minima even before the point S in figure 3, which therefore no more represents a spinodal point. In other words, the $U(2)$ model with $B \neq 0$ exhibits coexisting phases for *all* values of $A$ before the antispinodal point. (Note, however, that in some applications, when the $b$-boson is assumed to carry negative parity, the cubic term is forbidden by parity conservation and the model has only a second-order phase transition.) The increase of dimension for $l > 0$ introduces new features, particularly the possibility that a minimum of $V$ in $\beta$ is only a saddle point of the global dependence. This situation (well-known from the $l = 2$ case, e.g.) still tends to cause some confusion.

## 6. Higher order interactions

In most applications of interacting boson models to physical systems, the Hamiltonian contains only one- and two-body interactions and thus our classification is valid and appropriate. There are few applications to nuclei and molecules in which interactions of order higher than two-body have been considered, in particular three- and four-body interactions. A classification of phase transitions in these cases is by far more complex. The evaluation of the energy functional is not a problem, and can be (and has been) dealt with algebraic manipulations. For example, in the case $l = 2$, the energy functional written in terms of the variables $\beta$ and $\gamma$ for a Hamiltonian up to $k$-body terms, is

$$V(\beta, \gamma) = \frac{1}{(1+\beta^2)^k} \sum_{i,j} a_{ij} \beta^{2i+3j} \cos^j 3\gamma, \tag{28}$$

where the ranges of integers $i$ and $j$ are bounded by $2i + 3j \leqslant 2k$. This energy functional has many parameters, $a_{ij}$, and only by setting some of the parameters equal to zero, a study has been attempted.

A similar situation occurs for the catastrophe classification, where it is known that for codimension higher than 5 the elementary classification becomes infinite [28]. We are planning further studies in this direction to see whether or not a classification of the phase structure of boson models with three-body interactions is possible.

## 7. Conclusions

In this paper, we have analysed the phase structure of a class of $U(n)$ boson models, where the constituents are scalar bosons, $s$, and non-scalar bosons, $b^{(l)}$, with $2l + 1 = f = n - 1$ components, and with a one- and two-body Hamiltonian subject to 3D rotational invariance for $n =$ even $\geqslant 4$ and 2D rotational invariance for $n =$ odd $\geqslant 3$. Several models of this class are of current interest in nuclear and molecular physics, but potential applications range also to other fields, such as hadronic physics, superconductivity etc.

The geometry of the space associated with these models is $\mathcal{R}_f$. We have introduced in this space hyperspherical coordinates, with a radius $\beta$ and angles $\theta_i$ ($i = 1, \ldots, 2l$) and considered in this paper only phase transitions in the 'deformation' variable $\beta$. A classification of phase transitions in the hyperangles remains to be done.

We have found that while for $l = 0, 2, 4, 6, \ldots$ ($f = 1, 5, 9, 13, \ldots$) the system can have both first- and second-order transitions, for all other values it has only second-order transitions in $\beta$. The critical exponent of the order parameter $\beta_0$ for the second-order transitions is always $\lambda = 1/2$. For $l \geqslant 4$ ($f \geqslant 9$), there is the possibility of having second-order transitions with critical exponent $\lambda = 1$.

Finally, we have shown that all phase-transitional structures discussed here are specific realizations of the cusp catastrophe in the theory of structurally unstable potentials. This is a direct consequence of the fact that one- plus two-body interactions cannot produce potential energy terms of power higher than four, $\beta^4$.

Since the $N \to \infty$ ground-state phase transitions in systems with a finite single-particle Hilbert space are completely determined by their classical limit, the present considerations establish an equivalence class valid across a large number of specific applications.

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